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Developments in Nambu mechanics

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Abstract. This paper describes the following results in Nambu mechanics: the definition of simple physical systems; the result of applying the Kálnay and Tascón theorem to the study of different groupings of the n phase space variables into an s-coordinate and an $(n-s)$-momentum; the study of the intrinsic geometry of the curve that solves the Nambu equations of motion by explicit construction of the local coordinate system; the construction of sets of Hamiltonians that generate the same set of differential equations; a way to construct canonical transformations; a study of the intrinsic geometry of a system known to have chaotic behaviour: the Lorenz model and the correspondence of an oscillating Nambu system as the classical analogue of a simple version of the Hubbard model for superconductivity.

1. Introduction

This paper describes a number of developments in Nambu mechanics that fall into two categories: the first one concerns structural aspects (in the form of very general results) while the second is devoted to a number of particular cases whose aim is to illustrate known theorems or to define specific systems (like the free particle, the harmonic oscillator and the like). One important point that is not considered in this paper is the physical meaning—or either their relation to measurable quantities in the laboratory—of the parameters that appear in the $(n-1)$ Hamiltonians that define a Nambu system in an n-dimensional phase space.

As a start it is convenient to recall that Nambu mechanics is a generalization of Hamiltonian mechanics whose most relevant feature is that it admits a phase space of even or odd dimension. The variables that span phase space are the coordinates of the vector $\boldsymbol{x}=(x_1, x_2, \ldots, x_n)$. The time evolution of a dynamical variable $F(x)$ in an n-dimensional phase space is specified by $(n-1)$ functions of the n variables $H_i(x)$, $i=1, 2, \ldots, n-1$—called the Hamiltonians of the system—through

$$\mathrm{d}F(x)/\mathrm{d}t=[F, H_1, \ldots, H_{n-1}]=\partial(F, H_1, \ldots, H_{n-1})/\partial(x_1, \ldots, x_n) \qquad (1.1)$$

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where the notation $[f_1, f_2, \ldots, f_n]$ defines a generalized bracket with n places and $\partial(\ldots)/\partial(\ldots)$ is a Jacobian of order n. From this definition it follows at once that the generalized bracket is antisymmetric and behaves as a derivative.

If any of the Hamiltonians is substituted in (1.1) in place of $F(x)$ it turns out that the bracket vanishes; then all $H_i(x)$ are constants of the motion. They correspond to the integral surfaces of the set of coupled differential equations that describe the time evolution of the coordinates x_i.

$$v_i = \mathrm{d}x_i/\mathrm{d}t = [x_i, H_1, \ldots, H_{n-1}]. \tag{1.2}$$

The curve that solves the set (1.2) corresponds to the intersection of all these surfaces. In this sense Nambu mechanics uses the maximal information necessary to solve a system of coupled differential equations; namely, all its integral surfaces. The algorithm specified by Nambu generates completely integrable systems. If it is assumed that a point in the curve represents a state of a physical system then by describing a solution as the intersection of the integral surfaces of (1.2) there is no information as to how the point moves along the curve. This is specified if another surface, now explicitly time-dependent, of the form $W(x, t) = C$ with C a constant, is added so that the specific motion is obtained as the intersection of the curve with the surface $W(x, t) = C$. This is the method described by Cohen (1975). A direct consequence of (1.2) is that the Liouville condition is automatically satisfied, explicitly

$$\sum_{i=1}^{n} \partial v_i/\partial x_i = 0. \tag{1.3}$$

Canonical transformations are defined as those changes of independent variables such that the Jacobian equals one; gauge transformations are defined as changes in the functions $H_i(x)$ to new functions, say, $G_j(x)$ such that the Jacobian

$$\partial(G_1, \ldots, G_{n-1})/\partial(H_1, \ldots, H_{n-1}) = 1. \tag{1.4}$$

These are the basic facts of Nambu mechanics that will be used in the sequel so that for further details the reader is referred to Nambu's paper (Nambu 1973) (in particular for alternative definitions of the time evolution law of dynamical variables). Other aspects are discussed in Cohen (1975), Cohen and Kálnay (1975), Ruggeri (1975, 1981), Hirayama (1977), Oliveira (1977), Kálnay and Tascón (1978), Kobussen (1978), Angulo *et al* (1984), Codriansky and Gonzalez (1987).

Now a brief summary of the main criteria and results of this paper will be exhibited. One of the important results in the study on Nambu mechanics is the Kálnay and Tascón (1977) definition of what is meant by a coordinate and its canonically conjugate momentum—see section 2 for a brief summary of this result. It is natural to consider that two different groupings of the n phase space coordinates into an s-coordinate and an $(n\text{-}s)$-momentum $(1 \leqslant s \leqslant n)$ define two different mechanical systems; with this in mind we ask whether this is in fact true or on the contrary that all groupings are equivalent. We say that two groupings are equivalent if they are connected by a canonical transformation, a gauge transformation or a combination of both. It is found that in three-dimensional phase space the two possible groupings are equivalent while in four dimensions, of the three possibilities there is one grouping that is not equivalent to the other two: it is the one in which the four coordinates are separated into a two-coordinate and a two-momentum which in the specific case considered below corresponds to two Hamiltonian doublets. The same feature repeats in higher dimensions. Another result is the definition of a singlet (Codriansky and Gonzalez 1987); one interesting result is that the condition to be satisfied so that in two-dimensional phase space the doublet can be considered as two singlets is that the function that describes the time evolution of the doublet and the function that describes the time evolution of the two singlets be the real and imaginary parts of an analytical function.

The next point that is considered is the definition of elementary systems like the free particle and the harmonic oscillator. The basic idea in the definition is to maintain the general structure of the differential equations that define these systems in Hamiltonian mechanics. In this spirit the harmonic oscillator is described by a system of coupled differential equations that has the v_i written as a linear combination of the x_i. For each case at least one set of Hamiltonians is exhibited and in the case where more than one set is found the relevant canonical and/or gauge transformation is computed. This result is similar to the one found in Lagrangian mechanics where different Lagrangians can give rise to the same equation of motion (Okubo 1980). It is tempting to say, in the light of this result, that a physical system is defined by its evolution equation and not by its Hamiltonians; this is the standpoint adopted in this paper. In this way it is possible to characterize a given system by the simplest set of Hamiltonians or by any adequate representative of the equivalence class of the sets that generate a given equation of motion.

Next the study of more specific points is attacked. Among these: in three-dimensional phase space the intrinsic geometry of the solution to (1.2) is written down in terms of the Hamiltonians that characterize the system. The basic assumption is that the geometry of phase space is Euclidean so that the well known Frenet–Serret equations are translated into the language of Hamiltonians. If the same case is studied in four-dimensional phase space it is found that two more functions—together with the curvature and torsion—have to be included. When studying harmonic systems—the so-called Nambu harmonic oscillator—it is found that whenever the dimension of phase space is odd there is one root of the secular equation that vanishes so that the oscillating system is, in fact, even-dimensional. Another point studied refers to a specific way of constructing transformations by using both the canonicity and the gauge conditions.

The paper is organized as follows: in section 2 the Kálnay–Tascón theorem (Kálnay and Tascón 1977) is summarized; section 3 deals with two-dimensional phase space; in section 4 the intrinsic geometry in dimension n is discussed; section 5 presents canonical transformations and in section 6 physical systems are studied, among them the Lorenz model and a specific version of superconductivity.

Notation and conventions: boldface letters denote vectors, the summation convention is used throughout, d_{ij} denotes Dirac's delta function, $\partial_i = \partial/\partial x_i$, the Kálnay–Tascón (1977) result will be called the KT theorem.

2. A short summary of the Kálnay–Tascón theorem

The KT theorem defines what has to be understood by momenta conjugate to a coordinate in Nambu mechanics. The construction is done in such a way that a Lagrangian need not be used nor the specific form of the bracket that generates the time evolution of a particular system. In order to achieve the generalization from the usual classical mechanics (non-constrained Lagrangian or Hamiltonian mechanics) the main idea is to construct a mathematical object that resembles the Poincaré integral invariant of classical mechanics. Explicitly the construction goes as follows: assume the dimension of the Nambu phase space is n and that the set of n variables is separated in m groups with k variables each so that $km=n$. Phase space consists of m k-plets. Next consider an integer s such that $1 \leqslant s \leqslant k$ so that in each particular k-plet a set of s (real or complex) functions of the n coordinates is formed; this set is called an s-coordinate. If to the s-coordinate it is possible to add a set of $(k-s)$ functions so that

$$J_k^s = \sum_{r=1}^{m} \int \mathrm{d}q_{r1} \ldots \mathrm{d}q_{rs} \, \mathrm{d}f_1^r \ldots \mathrm{d}f_{k-s}^r \tag{2.1}$$

satisfies

$$\mathrm{d}J_k^s/\mathrm{d}t = 0 \tag{2.2}$$

where r labels each multiplet, the qs are the functions that define the s-coordinate and the fs define the so called $(k-s)$ Poincaré-momentum. Of course the above theorem remains true if each q is identified with a particular coordinate. This particular case follows from the fact that the Liouville condition is automatically satisfied in Nambu mechanics—at least in the version that is being used in the present paper. In view of this result in each of the multiplets a particular group of s coordinates is an s-coordinate and the remaining $(k-s)$ is the $(k-s)$ Poincaré momentum. In the cases considered in the present paper only one multiplet is present so that all examples are in fact a very particular case of this theorem. For further details see Kálnay and Tascón (1977).

3. Two-dimensional phase space

Consider two-dimensional phase space and denote the independent variables by $x = (x_1, x_2)$. If the variables are required to define a Hamiltonian doublet then one is chosen as the coordinate and the other as the canonically conjugated momentum. The time evolution of x_1 and x_2 is given in terms of the Hamiltonian $H(x)$ in the usual way

$$v_1 = \mathrm{d}x_1/\mathrm{d}t = \partial_2 H \qquad v_2 = \mathrm{d}x_2/dt = -\partial_1 H. \tag{3.1}$$

If, on the other hand, they are considered as a pair of singlets the evolution equations are different since in this case the relevent bracket is not the usual Poisson bracket but a symmetric bracket. In fact, as has been defined in Codriansky and Gonzalez (1987) the time evolution is computed in terms of a function $G(x)$ as

$$S_1 = \mathrm{d}x_1/\mathrm{d}t = \partial_1 G \qquad S_2 = \mathrm{d}x_2/\mathrm{d}t = \partial_2 G. \tag{3.2}$$

If it is now required that $v_1 = S_1$ and $v_2 = S_2$ the following equations are obtained

$$\partial_2 H = \partial_1 G \qquad -\partial_1 H = \partial_2 G \tag{3.3}$$

which are the Cauchy Riemann conditions for the real and imaginary parts of the analytic function $J = H + \mathrm{i}G$. If this condition is fulfilled then both descriptions are equivalent. This means that a given physical system can be considered either as a Hamiltonian doublet or as a pair of Nambu singlets. It is worthwhile to notice that the Liouville condition is automatically satisfied in the case of the Hamiltonian doublet while it is a limitation on G; only those G that are harmonic functions are candidate to satisfy (3.3)—then the companion H is also a harmonic function as is evident from (3.3). The two singlets are not independent but are in interaction as follows from the evolution equations. Also, while H is a constant of the motion G is not as can be easily checked from (3.1) and (3.2). An extremely simple example is $G(x) = -2a\ x_1 x_2$,

$H(x) = a(x_1^2 - x_2^2)$, where a is an arbitrary real constant, whose general solution is $x_1 = x_2 = A \exp(-2at)$, A constant.

As a preparation for future developments it is easy to write the Frenet–Serret equations that describe the way in which the local coordinate system defined by the unit tangent and normal vectors varies with time—only the system described by (3.1) is considered. Call $\boldsymbol{T}$ the unit tangent vector. The Frenet–Serret equations are then

$$dT/dt = kN \qquad dN/dt = -kT \tag{3.4}$$

where N is the normal to the curve and $k = k(x)$ is the curvature. If use is made of the evolution equations then it is found that k is completely determined once the Hamiltonian is fixed; in fact, from (3.1) the unit tangent vector is fixed by the Hamiltonian and from the definition of the unit normal vector N it is found that its explicit expression is fixed by H. Since $k = k(x)$ is defined as the norm of the vector (v_1, v_2) it follows that H determines completely the intrinsic geometry of the curve that satisfies (3.1). The expression for the curvature is

$$\begin{aligned} v^2k^2 = {} & (\partial_{11}H)^2(\partial_2 H)^4 + (\partial_{22}H)^2(\partial_1 H)^4 \\ & + 2[2(\partial_{12}H)^2 + \partial_{11}H\partial_2^2 H](\partial_1 H)^2(\partial_2 H)^2 \\ & - 4\partial_1 H\partial_2 H\partial_{12}H[(\partial_{11}H)(\partial_2 H)^2 + (\partial_1 H)^2(\partial_{22}H)] \end{aligned} \tag{3.5}$$

where

$$v^2 = (\partial_1 H)^2 + (\partial_2 H)^2. \tag{3.6}$$

As a last comment a canonical transformation is one to new coordinates y_1, y_2 with unit Jacobian. As a particular case, if the transformation is linear it is an element of SO(2). A gauge transformation reduces in this case to a simple addition of an arbitrary constant to H, the KT theorem is trivial in this case since a singlet is not considered in it. Finally, the only possibilities that can be considered in two-dimensional phase space are (i) a pair of singlets and (ii) one coordinate and its momentum canonically conjugated—which is, of course, the ordinary Hamiltonian doublet.

4. The intrinsic geometry in dimension n

Consider n-dimensional phase space. The independent variables are $\boldsymbol{x} = (x_1, \ldots, x_n)$ and the evolution equations are

$$v_i = \mathrm{d}x_i/\mathrm{d}t = \partial(x_i, H_1, H, \ldots, x_n) \tag{4.1}$$

the ith component of the tangent vector is defined as

$$T_i = v_i/v \tag{4.2}$$

where v is the norm of the vector $(v_1, \ldots, v_n)$, $v^2 = \sum v_i^2$. Next define the normal vector in the usual way as the vector N with components N_i given by

$$T N_i = \mathrm{d}T_i/\mathrm{d}t, \tag{4.3}$$

where $T = \sum T_i^2$.

Following the above procedure a set of n vectors is constructed in such a way that all are of unit length and each one orthogonal to the other $(n-1)$—the construction can fail if a constant vector is obtained in any of the $(n-2)$ first steps. It is easily seen that all vectors are expressed completely in terms of the $(n-1)$ Hamiltonians that define the system of coupled differential equations. Now the Frenet–Serret equations can be derived without difficulty. To this end it is necessary to consider the local coordinate system for two very near values of time and compare them. The result is a set of equations that describe the variations of the unit vectors when moving from one point to a neighbouring one. Denote the n vectors as V_i, $i=1, \ldots, n$ and its components as $V_i(k)$, $k=1, \ldots, n$. Then

$$\mathrm{d}V_i/\mathrm{d}t=\sum a_{ik}V_k \tag{4.4}$$

where the coefficients a_{ik} are given by

$$a_{ik}=(1-d_{ik})b_{ik} \tag{4.5}$$

with $d_{ik}=0$ if $i\neq k$ and $d_{ik}=1$ if $i=k$. Now $V_i \cdot V_j=0$ implies

$$a_{ik}=-a_{ki} \tag{4.6}$$

a relation that fixes the general structure of equations (4.4). A particular case are the relations in (3.4) for $n=2$. However, because of the particular form of (4.4) for $i=1$ other relations appear; in fact, for $i=1$: $a_{12}=$curvature and $a_{ik}=0$ if $k\neq 2$. This implies $a_{k1}=0$ if $k\neq 2$. With this, all results have been obtained to describe the local coordinate system.

In order to complete the description of the intrinsic geometry of a solution to a Nambu problem it is necessary to express all geometric quantities in terms of the Hamiltonians of the system. The arc length is

$$\mathrm{d}s^2=\sum_{i=1}^{n} v_i^2\,\mathrm{d}t^2=\sum_{i=1}^{n}[x_i, H_1, \ldots, H_{n-1}]^2\,\mathrm{d}t^2 \tag{4.7}$$

which can be used (if convenient) as the parameter that labels points on the curve. Since time is not the arc length—except in very special cases—all expressions will be written in terms of time derivatives which are computed as functions of the Hamiltonians. In this way all the geometry is related to the H_i; more precisely, the $(n-1)$ Hamiltonians define $(n-1)$ vectors at each point of the curve—the normals to the surfaces $H_i=C_i$, $i=1, \ldots, n-1$—to which another vector can be added that is orthogonal to all of them. After orthonormalizing this set, the local coordinate system is completely defined and can be related to the vectors that appear in the Frenet–Serret equations by an orthogonal transformation that is a function of the particular point of the curve. This means that the set of vectors constructed from the normals to the integral surfaces may not evolve in time according to the Frenet–Serret equations.

Remark 4.1. In Hamiltonian mechanics it is possible to proceed as in the Nambu case defining the tangent vector, the normal and so forth until the Frenet–Serret equations are obtained in a phase space of dimension $2k$. The main difference between the Nambu and Hamilton cases lies in the possibility of constructing the local coordinate system using the normals to the integral surfaces in Nambu mechanics while in Hamiltonian mechanics there are not enough known integral surfaces.

Remark 4.2. Up to this point it has been assumed that the underlying geometry of phase space is Euclidean. This assumption can be relaxed to consider a geometry defined by an arbitrary metric tensor $g_{ij}(x)$; then (4.7) is changed to

$$\mathrm{d}s^2=\sum g_{ij}v_iv_j\,\mathrm{d}t^2=\sum g_{ij}[x_i, \ldots, H_{n-1}][x_j, \ldots, H_{n-1}]\,\mathrm{d}t^2 \tag{4.8}$$

which implies that all expressions have to be modified accordingly. In particular, whenever scalar products appear, the metric tensor should be included. Thus, the definition of curvature will now be

$$k^2 = k(x)^2 = \sum g_{ij} v_i v_j \tag{4.9}$$

and similarly for all other quantities. It is an interesting problem to relate the underlying geometry to the properties of the Hamiltonians. This will not be touched here.

Remark 4.3. The system of equations (4.4) that define the Frenet–Serret equations can be considered as a Nambu system in the variables V_i (we thank I Cohen for calling our attention to this point) if the Liouville condition is satisfied. Now phase space is of dimension n^2. Since the coefficients a_{ik} depend on x it is necessary to eliminate them as functions of the $V_i(k)$; once this has been achieved, the system can be considered as a proper Nambu system if the Liouville condition is satisfied. To write the x_is in terms of the V_{ik}s it is enough to consider a group of n functions out of the n^2 available—call them $K_i(x)$, $i = 1, \ldots, n$—such that $\partial(K_1, \ldots, K_n)/\partial(x_1, \ldots, x_n) \neq 0$; in this case the inversion is possible. Returning to the system (4.4) it is easily verified that the Liouville condition is not satisfied automatically. In the present case this condition takes the form $B_{kl} V_{kl} = 0$ where $B_{kl} = \partial a_{ki}/\partial V_{il}$. The vanishing of the above expression ensures that the Nambu scheme is applicable. However, to find the Hamiltonians in the general case is an almost impossible task because of the unknown nature of the functions $a_{ik}(V_1, \ldots, V_n)$. If it happens that the a_{ik} are constants then the set of Hamiltonians is easily known: one of the Hamiltonians is quadratic while all the others are linear in the V_i. This remark is important in the sense that the Frenet–Serret equations do not introduce additional algebraic structures in the study of the Nambu system.

4.1. *The particular case $n = 3$*

In this subsection the explicit expressions for the unitary vectors, the curvature and torsion for $n = 3$ are presented. The Nambu equations for this case are

$$v = \mathrm{d}x/\mathrm{d}t = \operatorname{grad} H_1 \times \operatorname{grad} H_2. \tag{4.10}$$

Using (4.10) the results are

$$T = R \operatorname{grad} H_1 \times \operatorname{grad} H_2 = R' \operatorname{grad} H_1 \times \operatorname{grad} H_2 \tag{4.11}$$

where R and R' are given by

$$R^{-2} = |\operatorname{grad} H_1|^2 |\operatorname{grad} H_2|^2 - (\operatorname{grad} H_1 \cdot \operatorname{grad} H_2)^2 \tag{4.12}$$

$$R'^{-1} = |\operatorname{grad} H_1|\ |\operatorname{grad} H_2| \sin\beta \tag{4.13}$$

and β is the angle between grad H_1 and grad H_2. The arc length ds is given by

$$\mathrm{d}s^2 = R^{-2}\mathrm{d}t^2 = R'^{-1}\mathrm{d}t^2 \tag{4.14}$$

the tangential acceleration is ($s' = \mathrm{d}s/\mathrm{d}t$)

$$\begin{aligned} s' s'' = \{ & [|\operatorname{grad} H_2|^2 (\operatorname{grad} H_1 \cdot \operatorname{grad}) \operatorname{grad} H_1 + |\operatorname{grad} H_1|^2 (\operatorname{grad} H_2 \cdot \operatorname{grad}) \operatorname{grad} H_2] \\ & - (\operatorname{grad} H_1 \cdot \operatorname{grad} H_2)[(\operatorname{grad} H_2 \cdot \operatorname{grad}) \operatorname{grad} H_1 \\ & + (\operatorname{grad} H_1 \cdot \operatorname{grad}) \operatorname{grad} H_2]\} \cdot (\operatorname{grad} H_1 \times \operatorname{grad} H_2) \end{aligned} \tag{4.15}$$

and the normal vector $\boldsymbol{N}$

$$\boldsymbol{N}=(|\mathrm{d}T/\mathrm{d}t|s')^{-1}\{\mathrm{grad}\, s'^2/2-\boldsymbol{v}\times\mathrm{curl}\,\boldsymbol{v}$$
$$-(\mathrm{grad}\, H_1\times\mathrm{grad}\, H_2)\cdot\boldsymbol{D}(\mathrm{grad}\, H_1\times\mathrm{grad}\, H_2)\} \qquad (4.16)$$

where $\boldsymbol{D}=[\mathrm{grad}(\ln(|\mathrm{grad}\, H_1|\,|\mathrm{grad}\, H_2|)+\cot\beta\,\mathrm{grad}\,\beta]$. Finally, the binormal is

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=(|\mathrm{d}T/\mathrm{d}t|s')^{-1}\boldsymbol{v}\times(\boldsymbol{v}\cdot\mathrm{grad})\boldsymbol{v} \qquad (4.17)$$

where all quantities are defined in terms of the Hamiltonians.

Example 1. Consider the well known problem of a charged particle moving in a uniform magnetic field $\boldsymbol{B}=(0,0,\mathrm{B})$ and uniform electric field $\boldsymbol{E}=(\mathrm{E},0,0)$. Then the equations for the components of the velocity satisfy a system of equations that can be cast in the form of a Nambu system (Razavy and Kennedy 1974, Steeb and Euler 1991) (a, b and g constants)

$$\mathrm{d}v_1/\mathrm{d}t=-2ab(v_2-g) \qquad \mathrm{d}v_2/\mathrm{d}t=2ab\,v_1 \qquad \mathrm{d}v_3/\mathrm{d}t=0 \qquad (4.18)$$

with Hamiltonians

$$H_1=a\,v_3 \qquad H_2=b[(v_2-g)^2+v_1^2]. \qquad (4.19)$$

If the intrinsic geometry of the curve that solves this system is described it is found that the torsion is zero while the curvature is constant. The explicit results are

$$k^2=\{v_1^2+(v_2-g)^2\}^{-1} \qquad t=0. \qquad (4.20)$$

The curve is a circle which is, of course, the known result.

Example 2. Consider the Lorenz model (Lorenz 1963, Steeb and Euler 1991) described by the system of differential equations

$$\mathrm{d}u_1/\mathrm{d}t=u_2-shu_1 \qquad (4.21)$$

$$\mathrm{d}u_2/\mathrm{d}t=-u_1u_3+u_1-hu_2 \qquad (4.22)$$

$$\mathrm{d}u_3/\mathrm{d}t=u_1u_2-bhu_3 \qquad (4.23)$$

where s, h and b are arbitrary parameters. It is shown by Steeb and Euler (1991) that the system (4.20)–(4.22) is integrable if $h=0$ and that in this case a pair of —at most— quadratic Hamiltonians is

$$2H_1=-u_1^2+u_2^2+u_3^2 \qquad 2H_2=-u_1^2+2u_3. \qquad (4.24)$$

From these Hamiltonians the following expressions are found for the curvature (k) and torsion (t)

$$k^2=S^{-1}\{u_1^2(1-u_3)^2+u_2^2(1-u_3-u_1^2)^2+(u_2^2+u_1^2(1-u_3))^2\}$$
$$-S^{-2}u_1^2u_2^2\{(1-u_3)(2-u_3)+u_2^2\}^2 \qquad (4.25)$$

$$t=k^2S^{-3/2}3u_1u_2[(1-u_3)^2+u_2^2]\{u_1^2(1-u_3)-u_2^2\} \qquad (4.26)$$

where $S=u_2^2+u_1^2(1-u_3)^2+u_1^2u_2^2$. It is seen that if $u_3>1$ the sign of the curly bracket in (4.26) is always negative while if $u_3<1$ the sign is not defined but depends on the specific values of u_1 and u_2. As a result the value $u_3=1$ separates different types of behaviour of the solution curves to (4.20)–(4.22). Chaotic behaviour of the system shows in the vicinity of $u_3=1$ since a tiny variation in the numerical values of the constants H_1 and H_2 and in the initial conditions implies very different behaviour of the solution.

Example 3. Consider four-dimensional phase space. The intrinsic geometry is described by four-vectors which satisfy the following Frenet–Serret equations

$$\mathrm{d}\boldsymbol{T}/\mathrm{d}t = k\boldsymbol{N} \tag{4.27}$$

$$\mathrm{d}\boldsymbol{N}/\mathrm{d}t = -k\boldsymbol{T} + t\boldsymbol{B} + r\boldsymbol{B}' \tag{4.28}$$

$$\mathrm{d}\boldsymbol{B}/\mathrm{d}t = -t\boldsymbol{B} + s\boldsymbol{B}' \tag{4.29}$$

$$\mathrm{d}\boldsymbol{B}'/\mathrm{d}t = -r\boldsymbol{N} - s\boldsymbol{B} \tag{4.30}$$

where it is seen that if $r=0$ and $s=0$ the three-dimensional Frenet–Serret equations are recovered. As usual in this case it is assumed that the underlying geometry is Euclidean. The intrinsic geometric behaviour is determined by the four functions $k=k(\boldsymbol{x})$, $t=t(\boldsymbol{x})$, $r=r(\boldsymbol{x})$ and $s=s(\boldsymbol{x})$ where $\boldsymbol{x}=(x_1, x_2, x_3, x_4)$.

As an extremely simple example consider the following group of three Hamiltonians: $2H_1=B(x_2^2+x_3^2)$, $2H_2=x_1^2$ and $H_3=x_1+x_2+x_3+x_4$. Then the curve is plane and the only non-zero function is the curvature. Its expression is

$$k^2 = 3B^2x_1^2(x_2^2+x_3^2)^2[2(x_2^2+x_3^2-x_2x_3)]^{-2}. \tag{4.31}$$

Example 4. The inverse problem is interesting. In this case the intrinsic functions are known and the Hamiltonians are required. It is possible to exhibit explicit solutions in very simple cases as for instance if the curve is confined to a plane; under these conditions one of the Hamiltonians is quadratic and all the others are linear in the independent variables.

5. Canonical transformations

This section describes a particular way to generate canonical transformations in Nambu mechanics. More general approaches have been studied elsewhere (Marin 1975) where the general form of the generating functions have been given. These results will not be described here since they are not directly related to the results that follow. The aim is to exhibit a procedure to generate a canonical transformation that uses simultaneously the fact that the Jacobian of the transformation from a set of coordinates to a new one is unity and the fact that in a gauge transformation the Jacobian of the transformation from a set of Hamiltonians to a new one has to be unity also. So consider the following situation: two sets of coupled differential equations are given each with the corresponding set of Hamiltonians that generate the system of equations by using the Nambu algorithm, then the transformation that connects both systems of equations is required. This transformation will be generated as follows: call one set of variables $\boldsymbol{x}$ and the corresponding Hamiltonians $H_i(\boldsymbol{x})$, $i=1, 2, \ldots, n$ and the second set $\boldsymbol{y}$ with Hamiltonians $G_j(\boldsymbol{y})$, $j=1, 2, \ldots, n$. Now to transform the variables $\boldsymbol{x}$ into the variables $\boldsymbol{y}$—or vice versa—set $H_i(\boldsymbol{x})=G_i(\boldsymbol{y})$ $i=1, 2, \ldots, n$; this expresses $x_1, x_2, \ldots, n_{n-1}$ on terms of the ys and at the same time ensures that the Jacobian of the G's with respect to the H's equals one so that it is a gauge transformation. The remaining equation is the condition that the Jacobian of the transformation has to be one; this forces the transformation to be canonical. If a solution is found to this system of equations it is said that both sets of coupled differential equations are canonically related.

6. Physical systems in Nambu mechanisms

In this section the definition of a free particle, a harmonic oscillator and other systems is presented. The basic criterion to define any one of these systems in Nambu mechanics is to retain the structure of the system of differential equations that is known from Hamiltonian mechanics. One point must, however, be clarified from the start when considering any particular case: the definition of which of the variables spanning phase space play the role of coordinates and which play the role of momenta must be specified. The first three subsections (6.1 to 6.3) present situations on three-dimensional phase space, the other is devoted to four dimensions.

6.1. The free particle

Consider three-dimensional phase space; in this case it is possible to choose, say, x_1 as the only coordinate and x_2 and x_3 as the momenta. Then the evolution equations are (Ω and β are non-zero constants)

$$\mathrm{d}x_1/\mathrm{d}t=\Omega x_2+\beta x_3 \qquad \mathrm{d}x_2/\mathrm{d}t=0 \qquad \mathrm{d}x_3/\mathrm{d}t=0 \tag{6.1}$$

which has the form of the evolution equations for a free particle in Hamiltonian mechanics in the sense that $\mathrm{d}x_1/\mathrm{d}t$ depends linearly in the momenta. One pair of Hamiltonians that give rise to these equations is

$$H(x_1, x_2, x_3)=(x_2^2+x_3^2)/2 \qquad G(x_1, x_2, x_3)=-\beta x_2+\Omega x_3. \tag{6.2}$$

It is also possible to think that there are two coordinates, say, x_1' and x_2' and only one momentum, x_3'; then the evolution equations are

$$\mathrm{d}x_1'/\mathrm{d}t=\Omega' x_3' \qquad \mathrm{d}x_2'/\mathrm{d}t=\beta' x_3' \qquad \mathrm{d}x_3'/\mathrm{d}t=0 \tag{6.3}$$

with the Hamiltonians

$$2H'(x_1', x_2', x_3')=x_3'^2 \qquad G'(x_1', x_2', x_3')=\beta' x_1'-\Omega' x_2'. \tag{6.4}$$

At this point it is possible to ask whether the system described having two coordinates and one momentum and the system that has one coordinate and two momenta are different or not. To answer it is necessary to look for a canonical transformation that connects both; if this transformation does not exist it is said that both possibilities give rise to different physical systems. It turns out that the transformation exists and is given by

$$kx_1=\Omega' x_1'+\beta' x_2' \tag{6.5}$$

$$k^2x_2=-\beta\beta' x_1'+\beta\Omega' x_2'+k\beta\Omega x_3' \tag{6.6}$$

$$k^2x_3=\Omega\beta' x_1'-\Omega\Omega' x_2'+k\beta x_3' \tag{6.7}$$

where $k^2=\Omega^2+\beta^2=\Omega'^2+\beta'^2$. The values of Ω' and β' are not arbitrary but at the same time for any Ω and β a pair (Ω', β') can be found so that both systems—(6.1) and (6.3)—are connected by a canonical transformation. As a result both separations are in fact one and the same and as is easily seen they are connected by an orthogonal transformation. Thus, the KT theorem allows the definition of only one grouping which means only one physical system, at least for this particular case.

6.2. *Generalization of the above result*

It is interesting to investigate whether in other cases the two possible groupings in three-dimensional phase space are connected by a canonical transformation. A partial answer will be given by looking at the case in which phase space is two-dimensional and the Hamiltonian is the sum of the kinetic plus potential energy: $H(p, q)=p^2/2m+V(q)$. The evolution equations are $\mathrm{d}p/\mathrm{d}t=-V'(q)$ and $\mathrm{d}q/\mathrm{d}t=p/m$; it is the structure of these equations that should be maintained when passing to three dimensions. Therefore, the evolution equations in case of two coordinates (x_1, x_2) and one momentum (x_3) will be assumed to be

$$\mathrm{d}x_1/\mathrm{d}t=\Omega x_3 \qquad \mathrm{d}x_2/\mathrm{d}t=\beta x_3 \qquad \mathrm{d}x_3/\mathrm{d}t=F(\partial_1 V; \partial_2 V) \tag{6.8}$$

where Ω and β are constants, $V=V(x_1, x_2)$, $\partial_i V$ is the partial derivative of V with respect to x_i and $F(.,.)$ is an arbitrary function. In case there are two momenta (x_2', x_3') and one coordinate (x_1'), the evolution equations will be set up as

$$\mathrm{d}x_1'/\mathrm{d}t=\Omega' x_2+\beta x_3 \qquad \mathrm{d}x_2'/\mathrm{d}t=f\,\mathrm{d}B/\mathrm{d}x_1' \qquad \mathrm{d}x_3'/\mathrm{d}t=g\,\mathrm{d}B/\mathrm{d}x_1' \tag{6.9}$$

where Ω', f and g are constants and $B=B(x_1')$. Now assume there is a linear transformation that connects both sets of variables in the form $x_j'=m_{jk}x_k$; then after computing the time derivative of both sides and using (6.8) and (6.9) it is found that if $m_{13}\neq 0$, $B(x_1')$ must be at most a quadratic function. In fact, take $j=2$, then

$$\mathrm{d}x_2'/\mathrm{d}t=f\,\mathrm{d}B/\mathrm{d}x_1'=(m_{21}\Omega+m_{22}\beta)x_3+m_{23}F(\partial_1 V; \partial_2 V). \tag{6.10}$$

Now, remembering that $\mathrm{d}B(x_1')/\mathrm{d}t=\mathrm{d}B(m_{11}x_1+m_{12}x_2+m_{13}x_3)/\mathrm{d}t$ and differentiating (6.10) with respect to x_3 it is found that

$$m_{13}\mathrm{d}^2B/\mathrm{d}x_1'^2=m_{12}\Omega+m_{22}\beta \tag{6.11}$$

which shows that B is at most a quadratic function of its argument if $m_{13}\neq 0$. Returning to (6.10), it is found that its left-hand side is linear in x_1 and x_2 so that if it happens that F is a linear function of its arguments, V is at most a quadratic function of its arguments. On the other hand, for any $V(x_1, x_2)$, F must be chosen so that $F(\partial_1 V; \partial_2 V)$ is a linear function of x_1 and x_2.

Since the only requirement found up to now is $m_{13}\neq 0$, it is possible to find an orthogonal transformation that relates both sets of evolution equations and therefore in this case both groupings lead to the same physical system. It may well be that this argument can be generalized to more general type of Hamiltonians (that are not of the kinetic plus potential energy type) but this will not be pursued here. The conclusion reached at this point is that in three-dimensional phase space the two groupings allowed by the Kálnay and Tascón (1977) theorem are in fact only one because there is a canonical transformation that connects both of them—at leat for the family of systems considered in this point.

6.3. *The harmonic oscillator*

The evolution equations of the Nambu harmonic oscillator will be such that the time derivative of any variable is a linear combination of all of them:

$$\mathrm{d}x_i/\mathrm{d}t=A_{ij}x_j \tag{6.12}$$

where A_{ij} are antisymmetric coefficients. In the general form (6.12) there is no guarantee that the solution will be oscillatory and certain conditions have to be imposed on the coefficients that follow from writing a solution in the form

$$x_i(t) = C_i \exp ibt \tag{6.13}$$

where the C_i and b have to be computed from (6.12). Oscillations will be present if the roots of the characteristic equation are all real—this corresponds to no damping. Once the characteristic equation is written down it is found that, whatever the values of the A_{ij} there is one vanishing root because the determinant of the A_{ij} is zero due to the structure of the Nambu equations. This result is valid whenever phase space is odd-dimensional.

A particularly interesting illustration of this result comes from a naive classical version of a simple Hubbard model for superconductivity (Yang 1989, Zhang 1990). This is a one-band model (Zhang 1990) defined by the Hamiltonian

$$H = -t \sum c^{\dagger}_{rs} c_{r's} + \text{h.c.} + U \sum n_{ru} n_{rd} - \mu \sum n_{rs} \tag{6.14}$$

where t and U are the parameters that define the model and μ is the chemical potential u stands for spin up, d for spin down, $c^{\dagger}_{rs}$ and c_{rs} are creation and annihilation operators for spin s ($=u$ or d) located at place r of the lattice, $n_{rs} = c^{\dagger}_{rs} c_{rs}$, the total number of electrons is N and M is the total number of lattice sites. The operators

$$J_- = \sum (-1)^r c_{ru} c_{rd} \qquad J_+ = J^{\dagger}_- \qquad J_0 = (N - M)/2 \tag{6.15}$$

satisfy the $SU(2)$ commutation relations and in this sense there is an underlying $SU(2)$ symmetry for the model. Moreover, the operators J_+, J_- and J_0 satisfy the following commutation relations with the Hamiltonian H

$$[H, J_{\pm}] = \pm (U - 2\mu) J_{\pm} \qquad [H, J_0] = 0. \tag{6.16}$$

It is this set of equations that will be put into correspondence with a classical Nambu triplet. Before doing this the main result of Zhang (1990) will be quoted: the collective modes show three eigenfrequencies: 0 and $\pm(U - 2\mu)$. This triplet of collective modes appear as poles in the Fourier transforms of the response functions. Now, the classical version of this quantum model will be taken as follows: the quantum operators $J_{\pm}$ and J_0 are put into correspondence with the variables that span three-dimensional phase space—$j_{\pm}$, j_0—and the classical evolution equation for the classical variables will be generated by the Nambu bracket for the triplet which will be put into correspondence with the quantum bracket through the law 'i times the quantum bracket—given by (6.16)—equals the classical Nambu bracket after replacement of the operators $J_{\pm}$ and J_0 by the phase space variables $j_{\pm}$ and j_0'. To implement this prescription a pair of Hamiltonians have to be determined; these are exhibited in (6.17). The identification is in fact the result stated by Nambu (1973) in his study of quantization of the generalized Hamiltonian scheme. It is easy to check that the quantum Nambu bracket $[J_+, J_-, J_0]$ is equal to the Casimir of $SU(2)$. On the other hand, if the Hamiltonians in the quantized model are taken as H and J_0 then $[H, J_0] = 0$ is equation (40) of Nambu (1973). Since H and J_0 can be diagonalized simultaneously, the linear relation between H and J_0 is valid in the weaker form of an equality of matrix elements. In this way all conditions stated by Nambu (1973) are satisfied in the particular case considered. For studies in the quantization of Nambu mechanics see Nambu (1973), Bialynicki-Birula and Morrison (1991) and Sahoo and Valsakumar (1992).

The classical Nambu system that corresponds to the quantum system just described will be constructed requiring that the classical variables evolve in time according to expressions similar to the ones that appear in the right-hand side of (6.16). This will be taken as the '... algebraic mapping of the relationships which characterize....' (Nambu 1973) but this time in the reverse order, from the quantum to the classical case. After this identification, the Nambu triplet (j_+, j_-, j_0) evolves in time with the two Hamiltonians

$$h = j_+ j_- + j_0^2 \qquad g = (U - 2\mu) j_0 \tag{6.17}$$

which reproduce, at the classical level, equations (6.16). As a result, the system is a Nambu oscillator in three-dimensional phase space with frequencies O and $\pm(U-2\mu)$. In this sense this is the classical version of the quantum single band Hubbard model summarized in this example (for details of the quantum case see Yang 1989 and Zhang 1990).

Remark 6.1. The case presented here that relates Nambu triplets with superconductivity is a second example of such a correspondence. In Angulo *et al* (1984) it has been shown that another version of superconductivity has a triplet as a substructure of the whole system.

6.4. *Groupings in four dimensions*

Only the simplest case will be presented: that of a free particle. The evolution equations are

$$v_1 = \mathrm{d}x_1/\mathrm{d}t = 0 \qquad v_2 = \mathrm{d}x_2/\mathrm{d}t = x_1 \tag{6.18}$$

$$v_3 = \mathrm{d}x_3/\mathrm{d}t = 0 \qquad v_4 = \mathrm{d}x_4/\mathrm{d}t = x_3 \tag{6.19}$$

with Hamiltonians $H = x_2x_3 - x_1x_4$, $F = x_1$, $G = x_3$. The system (6.18), (6.19) describes a Nambu quadruplet that can be interpreted as two Hamiltonian doublets. The conditions under which this can be done depend on the existence of a Hamiltonian that performs the following task: start with two Hamiltonian doublets and call h_1 and h_2 the Hamiltonians that describe the time evolution of each doublet. Now a third function—h_3—is required in such a way that the two doublets form a Nambu quadruplet. For further details see Perez (1985). A second grouping describes a Nambu system in which there is a one-momentum and a three-coordinate. The evolution equations are in this case

$$V_1 = \mathrm{d}y_1/\mathrm{d}t = 0 \qquad V_i = a_i y_1 \tag{6.20}$$

where the a_i's, $i = 2, 3, 4$ are constants. The Hamiltonians that generate the system (6.20) are $h = y_1y_2 - a_2y_1y_4/a_4$, $f = a_3y_1$, $g = a_4y_3/a_3 - y_4$. Now a canonical transformation will be looked for so that both systems of evolution equations are related. The transformation is written

$$y_i = f_i(x) \tag{6.21}$$

where the f_i, $i = 1, 2, 3, 4$ are functions to be determined so that after using (6.18) and (6.19) the equations (6.20) are satisfied and moreover, the Jacobian must be unity. The resulting equations for the f_i are

$$x_1\partial f_1/\partial x_2 + x_3\partial f_1/\partial x_4 = 0 \qquad x_1\partial f_i/\partial x_2 + x_3\partial f_i/\partial x_4 = a_i f_1 \qquad i = 2, 3, 4. \tag{6.22}$$

After simple manipulation of the second equations in (6.22) it is readily found that the only differential equation that has to be studied is

$$x_1\partial F/\partial x_2 + x_3\partial F/\partial x_4 = 0 \tag{6.23}$$

where F is either f_1 or any of the combinations $f_i/a_i - f_j/a_j$, $i, j = 2, 3, 4$. The general solution to (6.23) is

$$F = [h_1(x_1, x_3, x_4) + x_2]/x_1 + [h_2(x_1, x_2, x_3) - x_4]/x_3 \tag{6.24}$$

where $h_1(x_1, x_3, x_4)$ and $h_2(x_1, x_2, x_3)$ are arbitrary functions. With this result at hand the differences $f_2 - f_3$, $f_2 - f_4$ and $f_3 - f_4$ are explicitly written down. From this it follows that the three functions f_1, f_2 and f_3 are not functionally independent so that the Jacobian of the transformation is zero. In this way it has been proved that both groupings cannot be connected by an invertible transformation—and, *a fortiori*, by a canonical one. This result shows that the grouping into two doublets, although a particular case of a Nambu quadruplet, remains separated from the other possible groupings—at least in four-dimensional phase space. In the spirit of the general formulation of the KT theorem it is possible to word this result as follows: it is not possible to find a set of functions in such a way that a system of two Hamiltonian doublets can be considered as a system with one momentum and three coordinates.

Remark 6.2. It is apparent from the above results that the Hamiltonian scheme is a particular case of the Nambu one since there is at least one example—four-dimensional phase space—in which the Nambu scheme is able to accommodate cases unrelated to the Hamiltonian one.

Remark 6.3. In the case that a transformation is sought to connect a system described by a one-coordinate and a three-momentum into a system described by a three-coordinate and a one-momentum it is easy to show that an orthogonal transformation performs the task. This is a simple example of a canonical transformation but since we are more interested in the existence of one canonical transformation rather than finding its most general form it is enough to exhibit one.

References

Angulo R, Codriansky S, Gonzalez-Bernardo C A, Kálnay A J, Pérez F, Rodriguez-Gomez J, Rodriguez-Nuñez J and Tello-Llanos R 1984 *Phys. Lett.* **104A** 106

Bialynicki-Birula I and Morrison P J 1991 *Phys. Lett.* **158A** 453

Codriansky S and Gonzalez Bernardo C A 1987 *Il Nuovo Cimento* **97** 64

Cohen I 1975 *Int. J. Theor. Phys.* **12** 69

Cohen I and Kálnay A J 1975 *Int. J. Theor. Phys.* **12** 61

Hirayama M 1977 *Phys. Rev.* D **16** 530

Kálnay A J and Tascón R 1977 *Int. J. Theor. Phys.* **16** 635

Kálnay A J and Tascón R 1978 *Phys. Rev.* D **17** 1552

Lorenz E 1963 *J. Atmos. Sci.* **20** 130

Marín F G 1975 Generating functions of a canonical transformation in Nambu's formalism. Solutions *Licenciado Thesis* (in Spanish) Universidad Central de Venezuela Caracas

Nambu Y 1973 *Phys. Rev.* D **7** 2405

Okubo S 1980 *Phys. Rev.* D **22** 919

Oliveira C 1977 *J. Math. Phys.* **18** 120

Perez F 1985 Studies in contemporary mechanics *MSc Thesis* (in Spanish) Instituto Venezolano de Investigaciones Cientificas Caracas

Razavy M and Kennedy F J 1974 *Can. J. Phys.* **52** 1532

Ruggeri G J 1975 *Int. J. Theor. Phys.* **12** 287

Ruggeri G J 1981 *Acta Científica Venezolana* **32** 203

Sahoo D and Valsakumar M C 1992 *Phys. Rev.* A **46** 4410

Steeb W H and Euler N 1991 *Il Nuovo Cimento* B **106** 263

Yang C N 1989 *Phys. Rev. Lett.* **63** 2144

Zhang S 1990 *Phys. Rev. Lett.* **65** 120